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Doklady Akademii Nauk SSSR, Vol 69, No 6,

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"Uniqueness of Solution of the Problem of Electrical Prospecting"

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/Note: The following report appeared in the regular 'Geophysics section of the thrice-monthly Doklady Akademii Nauk SSSR, Volume 69, No 6 (21 Dec 1949), pages 797-800./

The Stationary field of an electric current generated by a point source, located at a point M_0 of the boundary of a conducting half-space $z \geqslant 0$, is determined by a function (potential) $u(x,y,z)$ satisfying the equation

$$\frac{\partial}{\partial x}\left(\sigma\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\sigma\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\sigma\frac{\partial u}{\partial z}\right)=0$$

(where sigma σ = conductivity of the medium) and the condition $\partial u/\partial z = 0$ at $z = 0$, $M \neq M_0$, and possessing at the point M_0 a singularity of the type

$$u(x,y,z) = q.\frac{1}{r_0} + \bar{u}(x,y,z) \qquad \left(q = \frac{1}{2\pi\sigma_0}\right)$$

Here $\sigma_0 = \sigma(M_0)$; r_0 is the distance of the point $M(x,y,z)$ from $M_0(x_0,y_0,z_0)$; and $\bar{u}$ is a function bounded at M_0 and regular at infinity.

The electric characteristics of a medium is often studied by measurement of the field of a point source (or of its derivatives, determined by apparent resistances) on the surface $z = 0$. The purpose of this work is to show that for laminated media ($\sigma = \sigma(z)$) the value of the superficial potential cannot correspond to various electric cross sections. For a laminated medium the equation for u has the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{1}{\sigma}\frac{\partial}{\partial z}\left(\sigma\frac{\partial u}{\partial z}\right)=0 .$$

1. In virtue of the cylindrical symmetry of the problem it is evident that $u = u(\rho, z)$. Let us discuss the auxiliary function

$$Z(z,\lambda) = \int_0^\infty u(\rho,z) J_0(\lambda\rho)\rho\, d\rho$$
$$= q\frac{e^{-\lambda z}}{\lambda} + \bar{Z}(z,\lambda), \qquad \left(\bar{Z} = \int_0^\infty \bar{u}(\rho,z) J_0(\lambda\rho)\rho\, d\rho\right)$$

where $J_0(\lambda, \rho)$ is a Bessel function of zero order and 1-st kind.

We shall show that this function satisfies the equation

$$\frac{1}{\sigma}\frac{\partial}{\partial z}\left(\sigma\frac{dZ}{dz}\right) - \lambda^2 Z = 0$$

and additional conditions $dZ/dz\Big|_{z=0} = -q$, $Z(\infty) = 0$. The integral determining the function $\bar{Z}$ converges uniformly. This follows from the asymptotic behavior of $J_0(\lambda\rho)$ and from

$$\frac{\partial}{\partial\rho}(\sqrt{\rho}u) = O(\rho^{-\frac{3}{2}}).$$

Hence it follows that $Z(z, \lambda)$ is a continuous function of $z (0 \leq z \leq \infty)$. It is easy to be convinced that $Z(\infty, \lambda) = 0$. The integral

$$\int_0^\infty \frac{\partial u(z,\rho)}{\partial z} \cdot J_0(\lambda\rho)\rho\, d\rho = -q e^{-\lambda z} + \int_0^\infty \frac{\partial\bar{u}}{\partial z} J_0(\lambda\rho)\rho\, d\rho$$

converges absolutely and uniformly. It follows therefore that $Z(z, \lambda)$ has a continuous derivative along z for all z for which $\sigma(z)$ is continuous. At points where $\sigma(z)$ is discontinuous, the product $\sigma(z)dZ/dz$ is continuous. It follows also from this formula that $dZ/dz\Big|_{z=0} = -q$ and that $dZ(\infty, \lambda)dz = 0$.

We shall show that the integral

$$I = \int_0^\infty \frac{1}{\sigma(z)}\frac{\partial}{\partial z}\left(\sigma\frac{\partial u}{\partial z}\right) J_0(\lambda\rho)\rho\, d\rho$$

converges uniformly. Let us transform the integral

$$I_{\rho_1,\rho_2} = \int_{\rho_1}^{\rho_2} \frac{1}{\sigma(z)}\frac{\partial}{\partial z}\left(\sigma\frac{\partial u}{\partial z}\right) J_0(\lambda\rho)\rho\, d\rho = \int_{\rho_1}^{\rho_2} \frac{1}{\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) J_0(\lambda\rho)\rho\, d\rho$$

by integration by parts:

$$I = \rho\frac{\partial u}{\partial\rho} J_0(\lambda\rho)\Big|_{\rho_1}^{\rho_2} - u\rho\frac{d}{d\rho}J_0(\lambda,\rho)\Big|_{\rho_1}^{\rho_2} + \int_{\rho_1}^{\rho_2} u\rho\lambda^2 J_0(\lambda\rho)\rho\, d\rho .$$

Taking into account the asymptotic orders of u, $\partial u/\partial\rho$, J_0 and J_1 as well as the uniform convergence of the integral determining Z, we are convinced that the integral I converges uniformly and that the derivative $\frac{1}{\sigma}\frac{d}{dz}(\sigma\frac{dZ}{dz})$ exists. Passing to the limit, $\rho_1 \to 0$ and $\rho_2 \to \infty$, we find that

$$\frac{1}{\sigma}\frac{d}{dz}(\sigma\frac{dZ}{dz}) - \lambda^2 Z = 0, \quad Z'(0,\lambda) = -q, \quad Z(\infty,\lambda) = Z'(\infty,\lambda) = 0.$$

2. Differentiating the equation for Z, we obtain

$$\sigma\frac{d}{dz}(\frac{1}{\sigma}\frac{dZ}{dz}) - \lambda^2 Z_1 = 0, \quad Z_1\big|_{z=0} = q\sigma_0, \quad Z_1\big|_{z=\infty} = 0,$$

where $Z_1 = -\sigma\, dZ/dz$. By substituting the new variable, we obtain

$$\frac{d^2 Z}{d\zeta^2} - \frac{\lambda^2}{\sigma^2}Z_1 = 0, \quad Z_1\big|_{\zeta=0} = q\sigma_0, \quad Z_1\big|_{\infty} = 0, \quad \zeta = \int_0^z \sigma\, dz .$$

These conditions uniquely determine Z_1. Indeed the existence of two different functions satisfying these conditions would mean that their difference $\bar{Z}_1$, not identically equal to zero, satisfies the same equation and the conditions $\bar{Z}_1\big|_{\zeta_0} = 0$, $\bar{Z}_1\big|_{\infty} = 0$. The function $\bar{Z}_1$, by virtue of the equation, cannot have positive maximums and negative minimums. Hence it follows that $\bar{Z}_1 \equiv 0$ identically. It follows at once from these considerations that $Z_1 > 0$ for all z, and that $\bar{Z}_1(\zeta)$ is a not an increasing function.

Let us consider the nonhomogenous equation

$$\bar{Z}'' - \frac{\lambda^2}{\sigma^2}\bar{Z} = -f, \quad \bar{Z}(0) = 0, \quad \bar{Z}(\infty) = 0 .$$

It follows from the same considerations that $\bar{Z} \geqslant 0$ if $f \geqslant 0$.

We shall show that if

$$\bar{Z}_i'' - \frac{\lambda^2}{\sigma_i^2}\bar{Z}_i = -f, \quad \bar{Z}_i(0) = 0, \quad \bar{Z}_i(\infty) = 0 \quad (i = 1, 2)$$

then it follows from $\sigma_1 \geqslant \sigma_2$ that $Z_1 \geqslant Z_2$. Indeed, we have

$$\overline{Z}'' - \frac{\lambda^2}{\sigma_1^2}\overline{Z} = -\left(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right)\overline{Z}_2 \leqslant 0, \quad \overline{Z}(0) = \overline{Z}(\infty) = 0,$$

where $\overline{Z} = \overline{Z}_1 - \overline{Z}_2$; therefore $\overline{Z} \geqslant 0$ or $\overline{Z}_1 \geqslant \overline{Z}_2$.

If the right term expresses the local function for which

$$\int_{z_0-\varepsilon}^{z_0+\varepsilon} f_e\, dz = 1$$

then we may perform the passage to the limit at $\varepsilon \to 0$, and within limits of Z_ε we shall obtain the function, $K(\zeta, \zeta_0)$, of the point source satisfying in zeta ζ the homogenous equation, while at the point ζ_0 we have:

$$\left.\frac{dK}{d\zeta}\right|_{\zeta_0-0}^{\zeta_0+0} = -1 .$$

It is evident that, for equations with various σ_i (i = 1,2), $\sigma_1 \geqslant \sigma_2$ we have for the source function the inequality $K_1(\zeta,\zeta_0) \geqslant K_2(\zeta,\zeta_0)$.

3. Basic theorem. If the function $Z(z,\lambda)$ is defined as the solution of the equation

$$\frac{d^2 Z}{d\zeta^2} - \frac{\lambda^2}{\sigma^2} Z = 0, \qquad Z(\infty) = 0,$$

where $\sigma(\zeta)$ are piecewise analytical functions, $\sigma(\zeta) \geqslant \sigma_0 > 0 \quad (0 \leq \zeta \leq \infty)$, then $\sigma(\zeta)$ is uniquely determined by the values of

$$R(\lambda) = Z'(0,\lambda)/Z(0,\lambda) .$$

In other words, identical values of $R_1(\lambda)$ and $R_2(\lambda)$ cannot correspond to different functions $\sigma_1(\zeta)$ and $\sigma_2(\zeta)$.

Let us assume that to some functions $\sigma_1(z)$ and $\sigma_2(z)$ correspond identical values of $R_1(\lambda) = R_2(\lambda) = R(\lambda)$. Let us normalize the functions $Z_i(z,\lambda)$ by setting $Z_i(0,\lambda) = 1$ (i = 1,2). The function $\overline{Z}(z,\lambda) = Z_1(z,\lambda) - Z_2(z,\lambda)$ satisfies the equation

$$\overline{Z}'' - \frac{\lambda^2}{\sigma_1^2}\overline{Z} = -F = -\lambda^2\left[\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right] Z_2 \quad (Z_2(z) \geqslant 0)$$

and the conditions $\bar{Z}(0,\lambda) = \frac{d\bar{Z}(0,\lambda)}{d\zeta} \equiv \bar{Z}(\infty,\lambda) \equiv 0.$

The function $\bar{Z}$ may be expressed in the form

$$\bar{Z}(\zeta,\lambda) = \lambda^2 \int_0^\infty \left(\frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}\right) Z_2(\zeta_0) K_1(\zeta,\zeta_0)\, d\zeta_0 .$$

Without limitation of generality we may consider $q(\zeta) = \frac{1}{\sigma_2^2} - \frac{1}{\sigma_1^2}$ differs from zero for values of ζ as small as desired. If it were not so and $q(\zeta) \equiv 0$, then for $0 \leq \zeta \leq \zeta_1$ it is evident that $\bar{Z}(\zeta,\lambda) \equiv d\bar{Z}(\zeta,\lambda)/d\zeta \equiv 0$ and the origin of reckoning of ζ should start from ζ_1*.

*[Note: The piecewise analyticity of $\sigma(\zeta)$ is assumed only in order to guarantee the sign-constancy of $q(\zeta)$ near $\zeta = 0$. The class of admissible functions could be transformed in such a way that the sign-constancy of $q(\zeta)$ can be properly utilized. Doubtlessly the necessity of this assumption is connected with the method of proof.]

For equations with constant coefficients we have:

$$K(\zeta,\zeta_0) = \frac{\sinh\frac{\lambda}{\sigma}\zeta \cdot e^{-\frac{\lambda}{\sigma}\zeta_0}}{\frac{\lambda}{\sigma} e^{-\frac{\lambda}{\sigma}\zeta_0}\left(\sinh\frac{\lambda}{\sigma}\zeta_0 + \cosh\frac{\lambda}{\sigma}\zeta_0\right)}, \text{ for } \zeta < \zeta_0 .$$

In particular, $K(\zeta,\zeta_0) \geqslant \frac{1}{\lambda}\sinh\frac{\lambda}{\sigma_0}\zeta \, \frac{\sigma_0}{2\cosh\frac{\lambda}{\sigma}\zeta_0}$ for $\zeta < \zeta_0$

We assume that σ_1 has a lower limit; i.e. that $\sigma_1 \geqslant \sigma_0$. Hence it follows that $K_1(\zeta,\zeta_0) \geqslant K(\zeta,\zeta_0)$.

Furthermore, from assumptions made in respect to σ_1 and σ_2 it follows that $q(\zeta)$ has a constant sign within an interval $0 \leq \zeta \leq \zeta_1^*$.

For the sake of definiteness, let $q(\zeta) > q_0 > 0$. In this case, we have

$$\bar{Z}(\zeta,\lambda) \geqslant q_0 Z_2(\zeta_1) \int_0^{\zeta_3} \lambda \cdot \sinh\frac{\lambda}{\sigma_0}\zeta \, \frac{\sigma_0}{\cosh\frac{\lambda}{\sigma_0}\zeta}\, d\zeta_0 - M Z_2(\zeta_1) \int_{\zeta_1}^{\infty} \lambda \sinh\frac{\lambda}{\sigma_0}\zeta \, \frac{\sigma_0\, d\zeta_0}{\cosh\frac{\lambda}{\sigma_0}\zeta_0} = \bar{F},$$

$$(\bar{F} = \bar{F}(\zeta,\lambda)).$$

where $|q(\zeta)| \leq M$ for $\zeta > \zeta_1$. But in the case of sufficiently large λ_0, we have:

$$q_0 \int_0^{\zeta_3} \frac{d\zeta_0}{\cosh\frac{\lambda}{\sigma_0}\zeta_0} > M \int_{\zeta_1}^{\infty} \frac{d\zeta_0}{\cosh\frac{\lambda}{\sigma_0}\zeta_0} \quad \text{for } \lambda \geqslant \lambda_0 .$$

Because $\bar{Z}(0,\lambda) = 0$, we obtain $dZ(0,\lambda)/d\zeta \geqslant dF(0,\lambda)/d\zeta > 0$, ~~what~~ which contradicts the asumption.

Coming back to the function $u(z,\rho)$ we see that if to various σ_1 and σ_2 correspond functions $u_i(\rho,z)$ $(i = 1,2)$ such that $u_1(0,\rho) = u_2(0,\rho) = f(\rho)$, then at $z = 0$ the following functions also

$$Z_i(0,\lambda) = \frac{q}{\lambda} + \bar{Z}_i(0,\lambda) = z(\lambda), \quad \bar{Z}_i(0,\lambda) = \int_0^\infty \left[u_i(\rho,0) - \frac{q}{\rho}\right] J_0(\lambda\rho)\rho\, d\rho$$

will be equal, which are determined from equations:

$$\frac{1}{\sigma}\frac{d}{dz}\left(\sigma\frac{dZ_i}{dz}\right) - \frac{\lambda^2}{\sigma^2}Z = 0, \quad \frac{d}{dz}Z_i(0,\lambda) = -q, \quad Z_i(\infty,\lambda) = 0.$$

Assuming $\sigma\, dZ/dz = -Z_i^{(1)}$, we see that $Z_i^{(1)}$ satisfies the equations of the basic theorem, taking the values of

$$R = \frac{dZ_i^{(1)}(0,\lambda)/d\zeta}{Z_i^{(1)}(0,\lambda)}$$

to be equal; hence it follows that $\sigma_1(z) = \sigma_2(z)$.

Submitted 21 October 1949.

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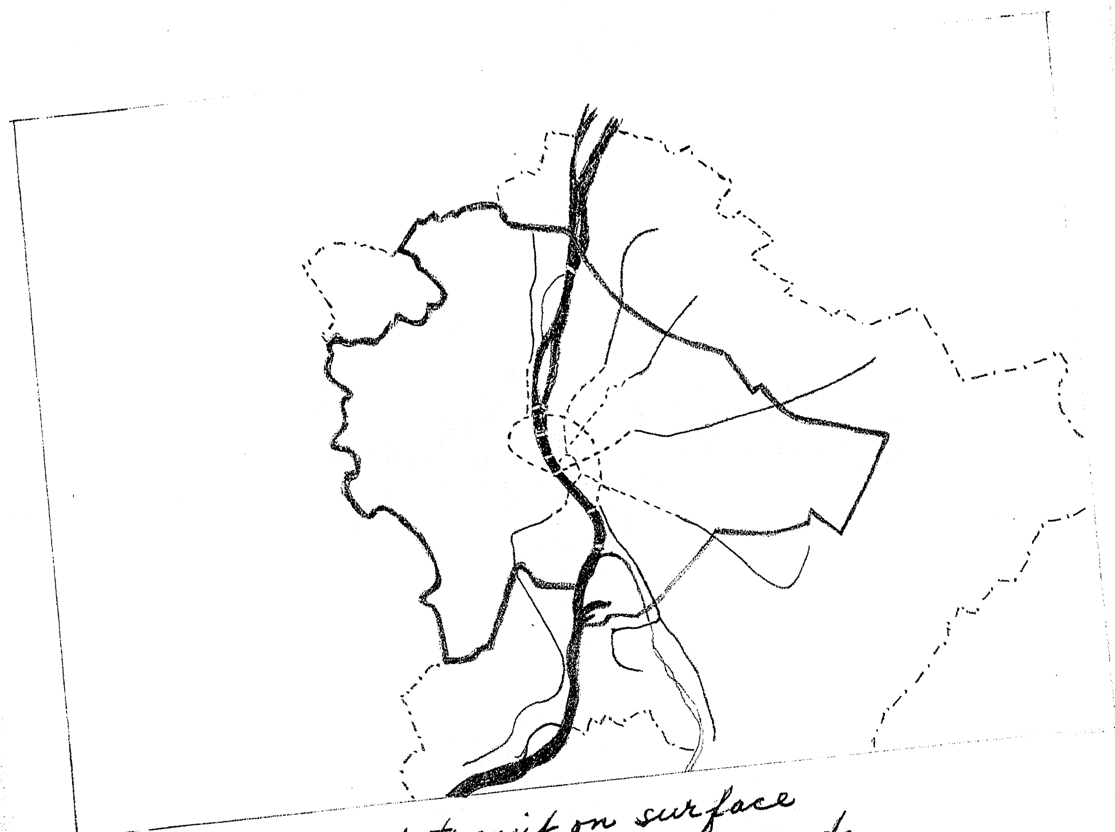

— rapid transit on surface
----- rapid transit underground